Taylor Domination, Turán lemma, and Poincaré-Perron Sequences

Dmitry Batenkov and Yosef Yomdin

Abstract We consider linear recurrence relations of the Poincaré type

$$a_k = \sum_{j=1}^{d} [c_j + \psi_j(k)] a_{k-j}, \quad k = d, d+1, \ldots, \quad \text{with } \lim_{k \to \infty} \psi_j(k) = 0.$$

We show that their solutions $a_0, a_1, \ldots, a_{d-1}, a_d, a_{d+1}, \ldots$ satisfy the Turán-like inequality: for certain N and R and for each $k \geq N+1$

$$|a_k| R^k \leq C \max_{i=0,\ldots,N} |a_i| R^i.$$

For the generating function $f(z) = \sum_{k=0}^{\infty} a_k z^k$ we interpret this last inequality as the "Taylor domination" property: all the Taylor coefficients of f are bounded through the first N of them. We give consequences of this property, showing that f belongs to the class of (s,p)-valent functions, which form a subclass of the classically studied p-valent ones.
We also consider moment generating functions, i.e. the Stieltjes transforms

$$S_g(z) = \int \frac{g(x)\,\mathrm{d}x}{1 - zx}.$$

We show Taylor domination property for such S_g when g is a piecewise D-finite function, satisfying on each continuity segment a linear ODE with polynomial coefficients. We utilize the fact that the moment sequence of g satisfies a Poincaré-type recurrence.

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1 Introduction

The problem of analytic continuation, i.e. of a global reconstruction of a function f and of its analytic properties, from the sequence of its Taylor coefficients at a point, is one of the most classical problems in Analysis. It took a central role in the classical investigations from the end of the nineteenth to the middle of the twentieth centuries. See [4, 5] and references therein for a sample of the classical results available. Some of these results are (arguably) considered as the deepest and most difficult results in Classical Analysis.

We are interested in a somewhat restricted setting of this problem: namely, the case when the sequence of Taylor coefficients depends on a finite number of parameters.

Definition 1.1. The sequence $\{a_k\}_{k=0}^{\infty}$ is said to satisfy a *linear recurrence relation of Poincaré type* (see [16, 18]) if

$$a_k = \sum_{j=1}^{d} [c_j + \psi_j(k)] a_{k-j}, \quad k = d, d+1, \ldots, \tag{1}$$

for some fixed coefficients $\{c_j, \psi_j(k)\}_{j=1}^{d}$, satisfying

$$\lim_{k\to\infty} \psi_j(k) = 0.$$

We also say that the coefficients $\{c_j, \psi_j(k)\}_{j=1}^{d}$ define an element $\mathcal{S}$ of the Poincaré class $\mathcal{R}$. If in addition these coefficients depend on a finite-dimensional parameter vector $\boldsymbol{\lambda} \in \mathbb{C}^m$, then we write $\mathcal{S}_\lambda$ for the corresponding recurrence.

Given a certain $\mathcal{S}_\lambda \in \mathcal{R}$, the solution of (1) is completely determined by λ and the initial data $\bar{a} = (a_0, a_1, \ldots, a_{d-1})$. In this case, we denote by $f(z) = f_{\lambda,\bar{a}}(z)$ the generating function of the corresponding solution $\{a_k\}_{k=0}^{\infty}$, i.e. the function represented in its disk of convergence by the power series

$$f_{\lambda,\bar{a}}(z) = \sum_{k=0}^{\infty} a_k z^k. \tag{2}$$

The main problem we discuss in this paper is the following.

Problem 1.1. To what extent is it possible to read out the global analytic properties of the function $f_{\lambda,\bar{a}}(z)$ from the parameters $\lambda, \bar{a}$? This includes, in particular, the position and type of singularities of $f(z)$, its ramification properties, and the bounds on the number of zeros of $f(z)$.

Our setting, described above, is motivated by two main long-standing open problems in qualitative theory of differential equations: Hilbert's 16th problem (2nd part) and Poincaré's Center-Focus problem. A promising approach

to these problems is based on the fact that the Taylor coefficients of the corresponding Poincaré first-return map satisfy a certain differential-type recurrence relation. The main question is essentially to obtain information on the zeroes of f. For more details, see e.g. [8].

Our approach stresses the recurrence relation (1) as the source of the infinite sequence of the Taylor coefficients. So, in contrast to the classical setting, we would like to read the finite-dimensional global information on f from a *finite-dimensional* input: the parameters $\lambda, \bar{a}$. While also in this setting most of the problems remain very difficult, we can hope to translate at least some of the classical results into the new setting.

In this paper we present a small piece of the above program, translating the classical results on univalent and multi-valent functions in [5, 13] into the language of the recurrence relation (1). These results were obtained mainly in connection to the classical Bieberbach conjecture, finally settled in [10]. The property of the Taylor coefficients, treated in [5, 13] is what we call "Taylor domination": bounding all the Taylor coefficients through the first few ones. See Section 2.

The classical Turán's lemma ([15, 22, 23]) is central in many problems in Harmonic Analysis. It provides *uniform* Taylor domination for rational functions (i.e. for solutions of the recurrence relation (1) with $\psi_j(k) \equiv 0$). In Section 3 we present a partial extension of the Turán lemma to general recurrence relations (1). As a consequence, we obtain a bound on the number of zeroes of the generating functions (2). In Section 4 we provide a natural setting of this result in terms of "(s,p)-valent functions". This notion provides a generalization of the classically considered multi-valent functions in [5, 13].

Not surprisingly, the problem of a full extension of the Turán lemma to general Poincaré type recurrence relations (1) is closely connected to the theorems of Poincaré and Perron and their extensions (see [6, 7, 14, 16, 17, 18, 19] and references therein), which provide an accurate asymptotic behavior of the solutions of (1). Using Perron's Second Theorem and its recent generalization by M.Pituk, we show that the solutions of (1), (even without the assumption on dependence on finite number of parameters), satisfy the Taylor domination property. Under additional assumptions we show uniform Taylor domination, which results in a bound on number of zeros. See Section 5 for details.

Sometimes (for instance in the first-order approximation of the Poincaré mapping to the Abel equation) the Taylor coefficients of the function f of interest are given as linear functionals, such as *moments*, of some other "simple" function g. In Section 6 we consider *moment-generating functions,* or *Stieltjes transforms* $f = S_g(z) = \int \frac{g(x)\mathrm{d}x}{1-xz} = \sum_{k=0}^{\infty} m_k z^k$, where $m_k = \int x^k g(x)\mathrm{d}x$ for g - piecewise D-finite functions, satisfying on each their continuity segment a linear homogeneous ODE with polynomial coefficients. In this case, the moment sequence of g satisfies a Poincaré-type recurrence $\mathcal{S}$ with $\psi_j\,(k)$-rational functions (in k), and we show Taylor domination property for S_g.

2 Taylor domination

2.1 Definitions and basic facts

Definition 2.1. A non-negative sequence $S(k) : \mathbb{N} \to \mathbb{R}_{\geqslant 0}$ is said to be of *sub-exponential growth* if for any $\varepsilon > 0$ we have

$$\lim_{k\to\infty} \frac{S(k)}{\exp(\varepsilon k)} = 0. \tag{3}$$

Let $f(z) = \sum_{k=0}^{\infty} a_k z^k,\ a_k \in \mathbb{C}$, be a power series with the radius of convergence $0 < \hat{R} \leqslant +\infty$.

Definition 2.2. Let a positive *finite* $R \leq \hat{R}$, a natural N, and a positive sequence $S(k)$ of sub-exponential growth, be fixed. The function f is said to possess an $(N, R, S(k))$ - Taylor domination property if for each $k \geq N+1$ we have

$$|a_k|R^k \leqslant S(k) \max_{i=0,\dots,N} |a_i|R^i.$$

For $S(k) \equiv C$ a constant we shall call this property (N, R, C)-Taylor domination.

Proposition 2.1. *$(N, R, S(k))$-domination implies (N, R', C)-domination for every $0 < R' < R$ and C a certain constant depending on $\frac{R'}{R}$ and on the sequence $S(k)$.*

Proof. Let $\rho \overset{\text{def}}{=} \frac{R'}{R} < 1$. First, let us show that the sequence $\rho^k S(k)$ is bounded:

$$\lim_{k\to\infty} \rho^k S(k) = \lim_{k\to\infty} S(k) \exp(k \ln \rho) = \lim_{k\to\infty} \frac{S(k)}{\exp\left(k \underbrace{\ln \rho^{-1}}_{>0}\right)},$$

and this is zero by (3).
So let

$$M \overset{\text{def}}{=} \sup_{k>0} \rho^k S(k).$$

Finally we have for $k > N$

$$|a_k| (R')^k = |a_k| R^k \rho^k \leqslant \rho^k S(k) \max_{i=0,\dots,N} |a_i| (R')^i \rho^{-i} \leqslant M \rho^{-N} \max_{i=0,\dots,N} |a_i| (R')^i .$$

This finishes the proof with $C = M\rho^{-N}$. $\square$

Lemma 2.1. *Let* $\{a_k\}_{k=0}^{\infty}$ *be a sequence satisfying*

$$\limsup_{k\to\infty} \sqrt[k]{|a_k|} = \sigma < +\infty, \tag{4}$$

and denote[1] *by* $\hat{R} \stackrel{def}{=} \sigma^{-1}$ *the radius of convergence of the power series*

$$f = \sum_{k=0}^{\infty} a_k z^k.$$

If $f \not\equiv 0$*, then for each finite and positive* $0 < R \leqslant \hat{R}$*, each* $N \in \mathbb{N}$ *satisfying* $a_N \neq 0$ *and for* $S(k)$ *defined as:*

$$S(k) \stackrel{def}{=} |a_k|\, \tilde{R}^k \times \left\{ \max_{i=0,\dots,N} |a_i|\, \tilde{R}^i \right\}^{-1}, \quad \textit{where } \tilde{R} \stackrel{def}{=} \begin{cases} \hat{R} & \sigma > 0 \\ R & \sigma = 0 \end{cases},$$

f *satisfies* $(N, R, S(k))$*-Taylor domination property.*

Proof. Consider two cases.

1. $\sigma > 0$. By Proposition 2.1, it is sufficient to show $\left(N, \hat{R}, S(k)\right)$-Taylor domination. We have for each $k > N$

$$|a_k|\, \hat{R}^k = S(k) \max_{i=0,\dots,N} |a_i|\, \hat{R}^i,$$

   and therefore the only thing left to show is that $S(k)$ is of subexponential growth. Denote

$$C \stackrel{\text{def}}{=} \max_{i=0,\dots,N} |a_i|\, \hat{R}^i > 0.$$

   Then we have for each $\varepsilon > 0$

$$\lim_{k\to\infty} \frac{S(k)}{\exp(\varepsilon k)} = \lim_{k\to\infty} \frac{|a_k|\, \hat{R}^k}{C \exp(\varepsilon k)} = \frac{1}{C} \lim_{k\to\infty} \left(\frac{|a_k|^{\frac{1}{k}}}{\sigma \exp(\varepsilon)} \right)^k.$$

   Denote $\eta = \exp(\varepsilon) > 1$. By (4), starting from some k_0 we have $\frac{|a_k|^{\frac{1}{k}}}{\sigma} < 1 + \frac{\eta - 1}{2} = \frac{1+\eta}{2}$, and therefore

$$\frac{|a_k|^{\frac{1}{k}}}{\sigma\eta} < \frac{1+\eta}{2\eta} = \frac{1}{2}\left(1 + \frac{1}{\eta}\right) < 1.$$

   As a result, the whole limit exists and is equal to zero.
2. If $\sigma = 0$, then for each $R < \infty$ we again have

[1] Just by applying the standard root test.

$$|a_k| R^k = S(k) \underbrace{\max_{i=0,\ldots,N} |a_i| R^i}_{=C},$$

and also for every $\varepsilon > 0$

$$\lim_{k\to\infty} \frac{S(k)}{\exp(\varepsilon k)} = \frac{1}{C} \lim_{k\to\infty} \left(\frac{|a_k|^{\frac{1}{k}} R}{\exp(\varepsilon)} \right)^k .$$

From the assumption, we actually have $\lim_{k\to\infty} |a_k|^{\frac{1}{k}} = 0$, and therefore the whole limit above exists and equals to zero.

This finishes the proof. □

As an immediate corollary we conclude that the Taylor domination property is satisfied by any converging power series in its maximal disk of convergence.

Corollary 2.1. *If* $0 < \hat{R} \leqslant +\infty$ *is the radius of convergence of* $f(z) = \sum_{k=0}^{\infty} a_k z^k$, *with* $f \not\equiv 0$, *then for each finite and positive* $0 < R \leqslant \hat{R}$, f *satisfies the* $(N, R, S(k))$*-Taylor domination property with* N *and* $S(k)$ *as defined in Lemma 2.1.*

As we show in the next subsection, the notion of Taylor domination is useful for investigating global analytic properties of the corresponding generating functions (in particular, their number of zeros), but only if the constants (N, R, C) are in a sense "as tight as possible". Therefore the main question related to Taylor domination is finding these optimal constants for certain families of generating functions $\{f_{\lambda,\bar{a}}\}$.

2.2 Global properties of generating functions and families

Applying the root test, we immediately conclude that the converse to Corollary 2.1 is also true.

Proposition 2.2. *Let* f *possess* $(N, R, S(k))$*-domination. Then the series*

$$\sum_{k=0}^{\infty} a_k z^k$$

converges in a disk of radius R^* *satisfying* $R^* \geqslant R$.

Taylor domination allows us to compare the behavior of $f(z)$ with the behavior of the polynomial $P_N(z) = \sum_{k=0}^N a_k z^k$. In particular, the number of zeroes of f can be easily bounded in this way.

Theorem 2.1 ([21, Lemma 2.2]). *There exists a finite function $M(N, \frac{R'}{R}, C)$, satisfying*

$$\lim_{\frac{R'}{R} \to 1} M = +\infty,$$

and equal to N for $\frac{R'}{R}$ sufficiently small, such that the following bound holds: if f possesses a (N, R, C) - Taylor domination property then for any $R' < R$, f has at most $M(N, \frac{R'}{R}, C)$ zeros in $D_{R'}$.
In particular, put $R_1 = \frac{R}{4}$, $R_2 = \frac{R}{2\max(C,2)}$ and $R_3 = \frac{R}{2^{3N}\max(C,2)}$. Then the number of zeroes of f in the disks D_{R_1}, D_{R_2} and D_{R_3} does not exceed $5N + \log_{5/4}(2+C)$, $5N + 10$ and N, respectively.

An explicit expression for M is given in [21, Proposition 2.2.2]. By Proposition 2.1, the corresponding bounds are true also in a general case of $(N, R, S(k))$ - Taylor domination.

It is important to stress that Taylor domination property is essentially equivalent to the bound on the number of zeroes of $f - c$, for each c. Let us give the following definition (see [13] and references therein).

Definition 2.3. A function f regular in a domain $\Omega \subset \mathbb{C}$ is called p-valent there, if for any $c \in \mathbb{C}$ the number of solutions in Ω of the equation $f(z) = c$ does not exceed p.

Theorem 2.2 ([5]). *If f is p-valent in the disk D_R of radius R centered at $0 \in \mathbb{C}$ then*

$$|a_k| R^k \leq (Ak/p)^{2p} \max_{i=0,\dots,p} |a_i| R^i.$$

In our notations, Theorem 2.2 claims that a function f which is p-valent in D_R, possesses an $(N, R, (Ak/p)^{2p})$ - Taylor domination property. Theorem 2.1 shows that the inverse is also essentially true.

For univalent functions, i.e. for $p = 1$, and for $a_0 = 0$, $R = 1$, the Bieberbach conjecture proved by De Branges in [10] claims that $|a_k| \leq k|a_1|$ for each k.

Let us now consider the families of generating functions (2), taking into account the dependence on the parameter vector λ. Our ultimate goal is to be able to bound the number of zeros of $f_\lambda(z)$ *uniformly in* λ. For this purpose, we introduce the notion of uniform Taylor domination, as follows.

Definition 2.4. Let $N \in \mathbb{N}$, $R(\lambda) : \mathbb{C}^m \to \mathbb{R}^+$ and a sequence $S(k) : \mathbb{N} \to \mathbb{R}^+$ of subexponential growth be fixed. The family $\{f_\lambda(z)\}$ of generating functions (2) is said to possess *uniform* $(N, R(\lambda), S(k))$*-Taylor domination property* if for each λ, the function $f_\lambda(z)$ possesses the $(N, R(\lambda), S(k))$-domination (c.f. Definition 2.2), with N and $S(k)$ *independent of* λ.

Consequently, if $\{f_\lambda(z)\}$ possesses uniform $(N, R(\lambda), S(k))$-Taylor domination, then for each λ the radius of convergence of $f_\lambda(z)$ is at least $R(\lambda)$, and the number of zeros in concentric disks $D_{R'}$ with a fixed proportion $\frac{R'(\lambda)}{R(\lambda)}$ can be uniformly (in λ) bounded, according to Theorem 2.1.

2.3 Rational generating functions and Turán's lemma

Let us start with the following well-known fact (see e.g. [11, Section 2.3]).

Proposition 2.3. *Consider a linear recurrence relation $\mathcal{S}$ with constant coefficients*

$$a_k = \sum_{j=1}^{d} c_j a_{k-j}, \quad k = d, d+1, \dots. \tag{5}$$

1. *For any initial data $\bar{a} = (a_0, a_1, \dots, a_{d-1})$, the general solution of (5) has the form*
$$a_k = \sum_{j=1}^{s} \sigma_j^k P_j(k), \tag{6}$$
where $\{\sigma_1, \dots, \sigma_s\}$ are the distinct roots of the characteristic equation
$$\Theta_{\mathcal{S}}(\sigma) = \sigma^d - \sum_{j=1}^{d} c_j \sigma^{d-j} = 0,$$
with corresponding multiplicities $\{m_1, \dots, m_s\}$, and each $P_j(k)$ is a polynomial of degree at most $m_j - 1$.
2. *The generating function $f_{\mathcal{S},\bar{a}}(z)$ of $\mathcal{S}$ is a rational function of the form*
$$f(z) = \frac{P(z)}{Q_{\mathcal{S}}(z)}, \quad \deg P(z) \leqslant d-1,$$
where
$$Q_{\mathcal{S}}(z) = z^d \Theta_{\mathcal{S}}\left(z^{-1}\right) = 1 - \sum_{j=1}^{d} c_j z^j$$
and $\deg P(z) \leqslant d-1$.
3. *Conversely, for each (regular at the origin) rational function $f(z) = \frac{P(z)}{Q(z)} = \sum_{k=0}^{\infty} a_k z^k$, with $Q(z) = \prod_{j=1}^{s} (1 - z\sigma_j)^{m_j}$ and $\sum_{j=1}^{s} m_j = d$, $\deg P(z) \leq d-1$, its Taylor coefficients a_k satisfy (5), with $\{c_j\}_{j=1}^{d}$ defined by*

$$Q(z) = 1 - \sum_{j=1}^{d} c_j z^j.$$

The most basic example of Taylor domination, concerning rational functions, is provided by the Turán lemma ([22, 23], see also [15]) as follows.

Theorem 2.3 ([22]). *Let* $\{a_j\}_{j=1}^{\infty}$ *satisfy the recurrence relation* (5)*. Using notations of Proposition 2.3, let* $\{x_1, \ldots, x_d\}$ *be the characteristic roots of* $\mathcal{S}$*, and put*

$$R \stackrel{def}{=} \min_{i=1,\ldots d} \left| x_i^{-1} \right|.$$

Then for each $k \geq d+1$

$$|a_k| \, R^k \leq C(d) \; k^d \max_{i=0,\ldots,d} |a_i| R^i. \tag{7}$$

This theorem provides a uniform Taylor domination (Definition 2.4) for rational functions in their maximal disk of convergence D_R, in the strongest possible sense. Indeed, after rescaling to D_1 the parameters of (7) depend only on the degree of the function, but not on its specific coefficients.

Turán's lemma can be considered as a result on exponential polynomials[2], and in this form it was a starting point for many deep investigations in Harmonic Analysis, Uncertainty Principle, Analytic continuation, Number Theory (see [15, 22, 23] and references therein).

The main problem we investigate in this paper is a possibility to extend a uniform Taylor domination in the maximal disk of convergence D_R, as provided by Theorem 2.3 for rational functions, to a wider class of generating functions of Poincaré type recurrence relations (1). As we show in Section 5, the $(N, R, S(k))$-domination is always satisfied in the entire disk of convergence for solutions of (1), however in general the domination is not uniform, as the sequence $S(k)$ depends on all the parameters $\{\psi_j\}_{j=0}^{\infty}$, $\bar{a}$ (in contrast to (7)).

We shall prove below in Section 3, for linear recurrence relations even more general than (1), a weaker version of Turán's lemma. It provides uniform Taylor domination not in the maximal disk of convergence, but in its concentric sub-disks of a sufficiently small radius.

[2] Indeed, the formula (6) shows that a_k are the values of the exponential polynomial $\phi(t) = \sum_{j=1}^{s} \lambda_j^t P_j(t)$ at the points $t \in \mathbb{N}$.

3 Uniform Taylor domination in a smaller disk

First let us define the class of recurrence relations slightly more general than the Poincaré class $\mathcal{R}$.

Definition 3.1. A linear recurrence relation $\mathcal{S}$ is said to belong to the class $\hat{\mathcal{R}}(K,\rho)$ (where $\rho > 0$ and $K > 0$ are two fixed parameters), if

$$a_k = \sum_{j=1}^{d} c_j(k) a_{k-j}, \quad k = d, d+1, \dots,$$

where the coefficients $c_j\left(k\right)$ satisfy for each $k \in \mathbb{N}$

$$\left|c_j\left(k\right)\right| \leqslant K\rho^j, \qquad j = 1, \dots, d.$$

So we do not require, as in (1), the convergence of the recurrence coefficients for k tending to infinity.

In this section we prove Taylor domination for solutions of the recurrence relations in $\hat{\mathcal{R}}\left(K,\rho\right)$. As we show later in Section 5, for each $\mathcal{S} \in \mathcal{R}$ there exist some $K, \rho > 0$ such that $\mathcal{S} \in \hat{\mathcal{R}}\left(K,\rho\right)$, and as a result we will obtain Taylor domination for such $\mathcal{S}$ with appropriately chosen constants.

Theorem 3.1. *Let $\left\{a_k\right\}_{k=0}^{\infty}$ be a solution of the recurrence relation $\mathcal{S} \in \hat{\mathcal{R}}\left(K,\rho\right)$. Put $R \stackrel{def}{=} \rho^{-1}$. Then for each $k \geqslant d$ we have*

$$\left|a_k\right| R^k \leqslant \left(2K+2\right)^k \max_{i=0,\dots,d-1} \left|a_i\right| R^i. \tag{8}$$

Proof. The proof is by induction on k. Denote

$$M \stackrel{\text{def}}{=} \max_{i=0,\dots,d-1} \left|a_i\right| R^i,$$
$$\eta \stackrel{\text{def}}{=} 2K+2,$$

and assume that

$$\left|a_\ell\right| R^\ell \leqslant \eta^\ell M, \quad \ell \leqslant k-1. \tag{9}$$

By definition, (9) holds for $0 \leqslant \ell \leqslant d-1$. We have

$$|a_k| R^k = R^k \Big|\sum_{j=1}^{d} c_j(k) a_{k-j}\Big| \leq K R^k \sum_{j=1}^{d} |a_{k-j}| \rho^j = K \sum_{j=1}^{d} |a_{k-j}| R^{k-j}.$$

By the inductive assumption $\left|a_{k-j}\right| R^{k-j} \leqslant \eta^{k-j} M$, therefore we conclude that

$$
\begin{aligned}
|a_k|\, R^k &\leqslant KM\sum_{j=1}^{d}\eta^{k-j} = KM\eta^{k-1}\sum_{s=0}^{d-1}\eta^{-s} \\
{\scriptstyle (\eta\,>\,2)} &\leqslant 2KM\eta^{k-1} \\
{\scriptstyle (\eta>2K)} &< \eta^{k} M.
\end{aligned}
$$

This finishes the proof. □

Corollary 3.1. *Let a recurrence $\mathcal{S} \in \hat{\mathcal{S}}\,(K,\rho)$ be given. Then the generating function $f_{\mathcal{S},\bar{a}}$ possesses a $\left(d-1, \frac{1}{\rho\eta}, \eta^{d-1}\right)$-Taylor domination property with $\eta = 2K+2$.*

Proof. Put $R' = \frac{1}{\rho\eta}$. Dividing (8) by η^k gives

$$
|a_k|\,(R')^k \leqslant \max_{i=0,\dots,d-1} |a_i|\,(R')^i\,\eta^i \leqslant \eta^{d-1} \max_{i=0,\dots,d-1} |a_i|\,(R')^i\,.
$$

This is precisely the definition of $\left(d-1, R', \eta^{d-1}\right)$ Taylor domination. □

Corollary 3.2. *Let a family $\{\mathcal{S}_\lambda\}$ of recurrences be given, such that for each $\lambda \in \mathbb{C}^m$ we have $\mathcal{S}_\lambda \in \hat{\mathcal{S}}\,(K, \rho\,(\lambda))$, with K independent of λ and $\rho\,(\lambda) : \mathbb{C}^m \to \mathbb{R}^+$ some positive function. Then the corresponding family of generating functions $\{f_{\lambda,\bar{a}}\}$ possesses a uniform $\left(d-1, \frac{1}{\eta\rho(\lambda)}, \eta^{d-1}\right)$-Taylor domination property, where $\eta = 2K+2$ is independent of λ.*

By the results above, we can now bound the number of zeros of $f_{\lambda,\bar{a}}(z)$ in any disk strictly inside the disk of radius $R'\,(\lambda) = \frac{R(\lambda)}{2K+2}$. In fact, as we shall see in the next section, much more can be deduced from the Taylor domination provided by Corollary 3.1.

4 (s,p)-valent functions

In this section we introduce the notion of an "(s,p)-valent function" which is a generalization of the classical notion of a p-valent function (see Definition 2.3 above and [13]).

The main reason to introduce this class is that for (s,p)-valent functions we can prove a kind of a rather accurate "distortion theorem" ([24]) which presumably is not valid in a larger class of p-valent functions (compare [13]). The distortion theorem shows that the behavior of f is controlled by the behavior of the polynomial with the same zeroes as f. This important property allows, in particular, to extend to (s,p)-valent functions the classical Remez inequality and its discrete versions (see [12, 20, 24, 25]).

Definition 4.1. A function f regular in a domain $\Omega \subset \mathbb{C}$ is called (s,p)-valent in Ω if for any polynomial $P(x)$ of degree at most s the number of solutions of the equation $f(x)=P(x)$ in Ω does not exceed p.

For $s=0$ we obtain the usual p-valent functions. Easy examples show that an (s,p)-valent function may be not $(s+1,p)$-valent. Let us fix integers p and $N \geq 10p+1$.

Lemma 4.1. *The function $f(x)=x^p+x^N,\ N \geq 10p+1$, is (s,p)-valent in the disk $D_{\frac{1}{3}}$ for any $s=0,\dots,p-1$, but only (p,N)-valent there.*

Proof. Taking $P(x)=x^p+c$ we see that the equation $f(x)=P(x)$ takes the form $x^N=c$ so for small c it has exactly N solutions in $D_{\frac{1}{3}}$. So f is (p,N)-valent in $D_{\frac{1}{3}}$, but not $(p,N-1)$-valent there.

To show that f is (s,p)-valent in the disk $D_{\frac{1}{3}}$ for any $s=0,\dots,p-1$ fix a polynomial $P(x)$ of degree $s \leq p-1$. Then the equation $f(x)=P(x)$ takes the form $-P(x)+x^p+x^N=0$. Applying to the polynomial $Q(x)=-P(x)+x^p$ of degree p (and with the leading coefficient 1) Lemma 3.3 of [24] we find a circle $S_\rho=\{|x|=\rho\}$ with $\frac{1}{3} \leq \rho \leq \frac{1}{2}$ such that $|Q(x)| \geq (\frac{1}{2})^{10p}$ on S_ρ. On the other hand $x^N \leq (\frac{1}{2})^{10p+1} < (\frac{1}{2})^{10p}$ on S_ρ. Therefore by the Rouché principle the number of zeroes of $Q(x)+x^N$ in the disk D_ρ is the same as for $Q(x)$, which is at most p. □

Our result below shows that generating functions $f_{\mathcal{S},\bar{a}}(z)$ of the recurrence relations $\mathcal{S}$ of the form (1) are (s,p)-valent, in appropriate domains, and for an appropriate choice of (s,p). Functions in many classes traditionally studied in Analysis have this form, in particular, solutions of linear ODE's with polynomial coefficients, so they are also (s,p)-valent.

Theorem 4.1. *Let $f(z)=f_{\mathcal{S},\bar{a}}(z)$ be a generating function of the recurrence relations $\mathcal{S} \in \hat{\mathcal{R}}(K,\rho)$, and let $R=\frac{1}{\rho},\ \hat{R}=\frac{R}{(2K+2)^d 2^{(s+d+1)}}$. Then for each $s \geq 0$ and $p=s+d+1$ the function $f(z)$ is (s,p)-valent in $D_{\hat{R}}$.*

Proof. By Definition 4.1 we have to show that for a given $s \geq 0$ and for any polynomial $P(x)$ of degree at most s the number of solutions of the equation $f(x)=P(x)$ in $D_{\hat{R}}$ does not exceed $p=s+d$. So we have to bound the number of zeroes of $f-P$. We shall do it via Taylor domination, provided by Theorem 3.1.

The first $s+1$ Taylor coefficients $\tilde{a}_0,\dots,\tilde{a}_s$ of $f-P$ may be arbitrary. However, for $k \geq s+1$ we have $\tilde{a}_k=a_k$. In particular, starting with $k=s+d+2$ the Taylor coefficients $\tilde{a}_k$ of $f-P$ satisfy recurrence relation $\mathcal{S}$:

$$\tilde{a}_k=\sum_{j=1}^{d} c_j(k)\tilde{a}_{k-j}, \quad k=s+d+2,s+d+3,\dots.$$

Now by Corollary 3.1 we conclude that starting with $k = s + d + 2$ we have

$$|a_k| \left(R'\right)^k \leq \ (2K+2)^{d-1} \max_{i=1,\dots,d} |a_{s+i+1}| \left(R'\right)^i,$$

with $R' = \frac{R}{2K+2}$. This is a $(s+d+1, R', (2K+2)^{d-1})$ - Taylor domination for $f - P$. Now application of Theorem 2.1 shows that $f - P$ has at most $s + d + 1$ zeroes in $D_{\hat{R}}$ for $\hat{R} = \frac{R'}{(2K+2)^{d-1}2^{(s+d+1)}} = \frac{R}{(2K+2)^d 2^{(s+d+1)}}$. $\square$

5 Taylor domination for Poincaré-type recurrences

Now we return to recurrence relations $\mathcal{S} \in \mathcal{R}$, i.e. of the form (1). The asymptotic behaviour of the solutions has been extensively studied, starting from Poincaré's own paper [18]. A comprehensive overview of the subject can be found in e.g. [11, Chapter 8]. The general idea permeating these studies is to compare the solutions of (1) to the solutions of the corresponding unperturbed recurrence relation (5).

Definition 5.1. Let $\mathcal{S} \in \mathcal{R}$ be given. The polynomial

$$\Theta_{\mathcal{S}}(\sigma) \stackrel{\text{def}}{=} \sigma^d - \sum_{j=1}^{d} c_j \sigma^{d-j}$$

is called the *characteristic polynomial* of $\mathcal{S}$. The roots of $\Theta_{\mathcal{S}}$ are called the *characteristic roots* of $\mathcal{S}$.

Let us briefly return to the constant-coefficient recurrences of Section 2.3. Let $\{\sigma_1, \dots, \sigma_d\}$ be the characteristic roots of $\mathcal{S}$. We have seen that for all initial data $\bar{a}$, the generating function $f_{\mathcal{S},\bar{a}}(z)$ is a rational function, regular at the origin, whose poles are a subset of $\left\{\sigma_1^{-1}, \dots, \sigma_d^{-1}\right\}$. Consequently, for some σ_j we have in fact

$$\limsup_{k \to \infty} \sqrt[k]{|a_k|} = |\sigma_j|.$$

O.Perron proved in [16] that this relation holds for a general recurrence of Poincaré type, but with an additional condition that $c_d + \psi_d(k) \neq 0$ for all $k \in \mathbb{N}$. In [17] M.Pituk removed this restriction, and proved the following result.

Theorem 5.1 (Pituk's extension of Perron's Second Theorem, [17]). *Let $\{a_k\}_{k=0}^{\infty}$ be any solution to a recurrence relation $\mathcal{S}$ of Poincaré class $\mathcal{R}$. Then either $a_k = 0$ for $k \gg 1$ or*

$$\limsup_{k \to \infty} \sqrt[k]{|a_k|} = |\sigma_j|,$$

where σ_j is one of the characteristic roots of $\mathcal{S}$.

This result, together with Corollary 2.1, implies the following.

Theorem 5.2. *Let $\{a_k\}_{k=0}^{\infty}$ be any nonzero solution to a recurrence relation $\mathcal{S}$ of Poincaré class $\mathcal{R}$ with initial data $\bar{a}$, and let R be the radius of convergence of the generating function $f_{\mathcal{S},\bar{a}}(z)$. Then necessarily $R>0$, and in fact $R=|\sigma|^{-1}$ where σ is some (depending on $\bar{a}$) characteristic root of $\mathcal{S}$. Consequently, f satisfies $(d-1,R,S(k))$-Taylor domination with $S(k)$ defined as in Lemma 2.1.*

Proof. The only thing left to show is that $a_m \neq 0$ for some $m=0,1,\dots,d-1$. Assume on the contrary that

$$a_0 = a_1 = \cdots = a_{d-1} = 0.$$

Plugging this initial data into the recurrence (1), we immediately conclude that $a_k=0$ for all $k\in\mathbb{N}$, a contradiction. □

Consider now the general setting, where $\mathcal{S}$ depends on a parameter vector $\lambda\in\mathbb{C}^m$. By Theorem 5.2, there is Taylor domination in the maximal disk of convergence, however the domination is not necessarily uniform in λ. In particular, no uniform bound on the number of zeros of $f_{\lambda,\bar{a}}(z)$ can be obtained in this way.

Accordingly, we pose the following question.

Problem 5.1. Does the "strong Turán inequality" (7), which holds, by Theorem 2.3, for solutions of (5), remain valid also for solutions of (1), in the maximal disk of convergence?

In the remainder of this section we provide some initial results in this direction.

Theorem 5.3. *Let $\{a_k\}_{k=0}^{\infty}$ satisfy a fixed recurrence $\mathcal{S}$ of the form (1). Let $\{x_1,\dots,x_d\}$ be the characteristic roots of $\mathcal{S}$, and put*

$$R \overset{def}{=} \min_{i=1,\dots d} \left|x_i^{-1}\right| > 0.$$

Let

$$\hat{N} \overset{def}{=} \min\left\{n: \quad \forall k>n: \ R^j \psi_j(k) \leqslant 2^d, \ j=1,\dots,d\right\},$$

and put $N=\hat{N}+d$. Then for each $k\geq N+1$ we have

$$|a_k| R^k \leq 2^{(d+3)k} \max_{j=0,\dots,N} |a_j| R^j.$$

Proof. Set $K \overset{\text{def}}{=} 2^{d+1}$ and $\rho \overset{\text{def}}{=} R^{-1} = \max_j |x_j|$. First let us show that $\mathcal{S} \in \hat{\mathcal{R}}(K, \rho)$. We have

$$Q_{\mathcal{S}}(z) = z^d \Theta_{\mathcal{S}}\left(z^{-1}\right) = \prod_{j=1}^{d} (1 - z x_i) = 1 - \sum_{j=1}^{d} c_j z^j.$$

Therefore for each $j = 1, \ldots, d$

$$|c_j| = |e_j(x_1, \ldots, x_d)| \leqslant \binom{d}{j} \rho^j \leqslant 2^d \rho^j,$$

where $e_j(\cdot)$ is the elementary symmetric polynomial of degree j in d variables. Thus we have for $k \geq N+1$

$$|c_j + \psi_j(k)| \leqslant |c_j| + |\psi_j(k)| \leqslant 2^d \rho^j + 2^d \rho^j = K \rho^j.$$

Now by Theorem 3.1, we have

$$\begin{aligned} |a_k| R^k &\leqslant (2K+2)^k \max_{i=0,\ldots,N} |a_i| R^i \leqslant \left(2^{d+2}+2\right)^k \max_{i=0,\ldots,N} |a_i| R^i \\ &< 2^{(d+3)k} \max_{i=0,\ldots,N} |a_i| R^i. \end{aligned}$$

This completes the proof. □

Corollary 5.1. *Under the conditions of Theorem 5.3, the generating function $f_{\mathcal{S},\bar{a}}(z)$ possesses (N, R', C) Taylor domination with $R' = 2^{-(d+3)} R$ and $C = 2^{(d+3)N}$.*

In particular, the corresponding bound on the number of zeros of $f_{\mathcal{S},\bar{a}}(z)$ holds in any concentric disk strictly inside $D_{R'}$.

Proof. Exactly as in the proof of Corollary 3.1. □

Theorem 5.3 provides a partial answer to Problem 5.1: Taylor domination in a smaller disk $D_{R'}$. One can consider two possible approaches to the extension of these results to the full disk of convergence D_R. First, asymptotic expressions in [6, 17] may be accurate enough to provide an inequality of the desired form. If this is a case, it remains to get explicit bounds in these asymptotic expressions.

Second, one can use a "dynamical approach" to recurrence relation (1) (see [9, 14, 19] and references therein). We consider (1) as a non-autonomous linear dynamical system T. A "non-autonomous diagonalization" of T is a sequence $\mathcal{L}$ of linear changes of variables, bringing this system to its "constant model" T_0, provided by the limit recurrence relation (5).

If we could obtain a non-autonomous diagonalization $\mathcal{L}$ of T with an explicit bound on the size of the linear changes of variables in it, we could obtain the desired inequality as a pull-back, via $\mathcal{L}$, of the Turán inequality for T_0.

5.1 Example of uniform domination

Now consider some family $\{\mathcal{S}_\lambda\} \subset \mathcal{R}$. Since the index N in Theorem 5.3 and Corollary 5.1 depends on $\left\{c_j^{(\lambda)},\ \psi_j^{(\lambda)}\right\}$, in general we would not obtain uniform Taylor domination. However, in some particular cases this is possible.

Example 5.1. Consider the family $\{\mathcal{S}_\lambda\} \subset \mathcal{R}$ with the following properties:

1. The constant parts $\{c_j\}_{j=1}^{d}$ do not depend on λ. Consequently, the characteristic polynomial $\Theta_{\mathcal{S}}(\sigma)$ and the radius of convergence R are also independent of λ.
2. The perturbations $\left\{\psi_j^{(\lambda)}(k)\right\}$ are uniformly bounded, i.e. for each $j = 1,\dots,d$ $\exists M_j \in \mathbb{N}$ s.t. $\forall k > M_j$ we have $\sup_\lambda \left|\psi_j^{(\lambda)}(k)\right| < +\infty$, and also
$$\lim_{k\to\infty} \sup_{\lambda} \left|\psi_j^{(\lambda)}(k)\right| = 0.$$

Then it is easy to see that there exists $\hat{N} \in \mathbb{N}$, independent of λ, such that for all $k > \hat{N}$, $j = 1,\dots,d$ and all $\lambda \in \mathbb{C}^m$ we have

$$R^j |\psi_j^{(\lambda)}(k)| \le 2^d.$$

Applying Theorem 5.3 and putting $N \stackrel{\text{def}}{=} \hat{N} + d$, we conclude that for each $k \geqslant N+1$

$$|a_k| R^k \le 2^{(d+3)k} \max_{j=0,\dots,N} |a_j| R^j,$$

for all solutions $\{a_k\}_{k=0}^{\infty}$ of all $\mathcal{S}_\lambda$ in the family. Consequently, the family $\{f_{\lambda,\bar{a}}(z)\}$ satisfies uniform (N, R', C) Taylor domination with $R' = 2^{-(d+3)}R$ and $C = 2^{(d+3)N}$.

Therefore, the main problem in obtaining uniform Taylor domination is choosing the "correct" subclasses of $\mathcal{R}$.

6 Piecewise D-finite functions

Recurrence relations of Poincaré type are intimately related with power series solutions to linear ODE's with meromorphic coefficients. In particular, the subclass of holonomic power series corresponds to solutions of (1), with $\psi_j(k)$ rational functions. In this section we investigate a certain subclass of holonomic power series, defined by the Stieltjes integral transforms

$$f(z) = S_g(z) = \int \frac{g(x)\,\mathrm{d}x}{1 - zx}, \tag{10}$$

where $g(x)$ belongs to the class of the so-called *piecewise D-finite functions* [1], which are essentially solutions of holonomic ODEs with finite number of discontinuities of the first kind.

Using the expansion $(1 - zx)^{-1} = \sum_{k=0}^{\infty} (zx)^k$ for $|z| < \frac{1}{|x|}$, we obtain the following useful representation of $S_g(z)$.

Proposition 6.1. *Let $g : [a, b] \to \mathbb{R}$ be bounded and integrable on $[a, b]$. Then the Stieltjes transform (10) of g is regular at the origin, and it is given by the moment-generaing function*

$$S_g(z) = \sum_{k=0}^{\infty} m_k z^k, \quad \textit{where } m_k \stackrel{def}{=} \int_a^b x^k g(x)\,\mathrm{d}x.$$

Definition 6.1. A real-valued bounded integrable function $g : [a, b] \to \mathbb{R}$ is said to belong to the class $\mathcal{PD}(\mathfrak{D}, p)$ if it has $0 \leqslant p < \infty$ discontinuities (not including the endpoints a, b) of the first kind, and between the discontinuities it satisfies a linear homogeneous ODE with polynomial coefficients $\mathfrak{D}\, g = 0$, where

$$\mathfrak{D} = \sum_{j=0}^{n} p_j(x) \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^j, \quad p_j(x) = \sum_{i=0}^{d_j} a_{i,j} x^i.$$

Obtaining uniform Taylor domination for S_g where g belongs to particular subclasses of $\mathcal{PD}$ can be considered our ultimate goal. In this paper we provide initial results in this direction. Our approach is based on the following result.

Theorem 6.1 ([1]). *Let $g \in \mathcal{PD}(\mathfrak{D}, p)$. Denote the discontinuities by $a = x_0 < x_1 < \cdots < x_p < x_{p+1} = b$. Then the moments $m_k = \int_a^b g(x)\,\mathrm{d}x$ satisfy*[3] *the recurrence relation*

$$\sum_{j=0}^{n} \sum_{i=0}^{d_j} a_{i,j} (-1)^j (i + k)_j\, m_{i-j+k} = \sum_{\ell=0}^{p+1} \sum_{j=0}^{n-1} x_\ell^{k-j} (k)_j\, c_{\ell,j}, \quad k = 0, 1, \ldots, \tag{11}$$

where each $c_{\ell,j}$ is a homogeneous bilinear form in the two sets of variables

$$\{p_m(x_\ell), p_m'(x_\ell), \ldots, p_m^{(n-1)}(x_\ell)\}_{m=0}^n,$$
$$\{g(x_\ell^+) - g(x_\ell^-),\ g'(x_\ell^+) - g'(x_\ell^-), \ldots, g^{(n-1)}(x_\ell^+) - g^{(n-1)}(x_\ell^-)\},$$

and $(a)_b = a(a-1) \times \cdots \times (a - b + 1) = \frac{\Gamma(a)}{\Gamma(b)}$ is the Pochhammer symbol for the falling factorial.

[3] For consistency of notation, the sequence $\{m_k\}$ is understood to be extended with zeros for negative k.

In what follows, we shall use some additional notation.
Denote for each $j=0,\dots,n$

$$\alpha_j \stackrel{\text{def}}{=} d_j - j.$$

Let $\alpha \stackrel{\text{def}}{=} \max_j \alpha_j$, and denote for each $\ell = -n,\dots,\alpha$

$$q_\ell(k) \stackrel{\text{def}}{=} \sum_{j=0}^{n} (-1)^j a_{\ell+j,j} (k+\ell+j)_j . \tag{12}$$

Furthermore, denote for $k=0,1,\dots$

$$\varepsilon_k \stackrel{\text{def}}{=} \sum_{\ell=0}^{p+1}\sum_{j=0}^{n-1} x_\ell^{k-j} (k)_j c_{\ell,j}.$$

Then the recurrence relation (11) can be written in the form

$$\sum_{\ell=-n}^{\alpha} q_\ell(k) m_{k+\ell} = \varepsilon_k, \quad k=0,1,\dots. \tag{13}$$

It is well-known that the sequence $\{\varepsilon_k\}_{k=0}^{\infty}$ satisfies a recurrence relation $\mathcal{S}$ of the form (5) with constant coefficients, whose characteristic roots are precisely $\{x_0,\dots,x_{p+1}\}$, each with multiplicity n. Let the characteristic polynomial $\Theta_{\mathcal{S}}(z)$ of degree $\tau \stackrel{\text{def}}{=} n(p+2)$ be of the form

$$\Theta_{\mathcal{S}}(\sigma) = \prod_{j=0}^{p+1} (\sigma - x_j)^n = \sigma^\tau - \sum_{i=1}^{\tau} b_i \sigma^{\tau-i}, \tag{14}$$

then

$$\varepsilon_k = \sum_{j=1}^{\tau} b_j \varepsilon_{k-j}, \quad k=\tau,\tau+1,\dots.$$

Rewrite this last recurrence as

$$\varepsilon_{k+\tau} = \sum_{j=0}^{\tau-1} b_{\tau-j}\varepsilon_{k+j}, \quad k=0,1,\dots. \tag{15}$$

Now denote the vector function $\mathbf{w}(k) : \mathbb{N} \to \mathbb{C}^{\alpha+n+\tau}$ as

$$\mathbf{w}\left(k\right) \stackrel{\text{def}}{=} \begin{bmatrix} m_{k-n} \\ \vdots \\ m_{k+\alpha-1} \\ \varepsilon_k \\ \vdots \\ \varepsilon_{k+\tau-1} \end{bmatrix}.$$

Then by (13) and (15) we see that $\mathbf{w}\left(k\right)$ satisfies the linear system

$$\mathbf{w}\left(k+1\right) = \left[\begin{array}{ccccc|ccccc} 0 & 1 & 0 & \dots & 0 & & & \mathbf{0}^{(\alpha+n-1)\times\tau} & & \\ 0 & 0 & 1 & \dots & 0 & & & & & \\ \dots & & & & & & & & & \\ -\frac{q_{-n}(k)}{q_\alpha(k)} & -\frac{q_{-n+1}(k)}{q_\alpha(k)} & \dots & & -\frac{q_{\alpha-1}(k)}{q_\alpha(k)} & \frac{1}{q_\alpha(k)} & 0 & \dots & & 0 \\ & & & & & 0 & 1 & 0 & \dots & 0 \\ & & & & & 0 & 0 & 1 & \dots & 0 \\ & & \mathbf{0}^{\tau\times(\alpha+n)} & & & \dots & & & & \\ & & & & & b_\tau & b_{\tau-1} & \dots & & b_1 \end{array}\right] \mathbf{w}\left(k\right). \tag{16}$$

Now we can show Taylor domination for the Stieltjes transform $S_g\left(z\right)$, utilizing the system version of Theorem 5.1.

Definition 6.2. The vector function $\mathbf{y}\left(k\right) : \mathbb{N} \to \mathbb{C}^n$ is said to satisfy a linear system of Poincaré type, if

$$\mathbf{y}\left(k+1\right) = \left(A + B\left(k\right)\right)\mathbf{y}\left(k\right), \tag{17}$$

where A is a constant $n \times n$ matrix and $B\left(k\right) : \mathbb{N} \to \mathbb{C}^{n\times n}$ is a matrix function satisfying $\lim_{k\to\infty} \|B\left(k\right)\| = 0$.

Theorem 6.2 ([17]). *Let the vector* $\mathbf{y}\left(k\right)$ *satisfy the perturbed linear system of Poincaré type* (17). *Then either* $\mathbf{y}\left(k\right) = \mathbf{0} \in \mathbb{C}^n$ *for* $k \gg 1$ *or*

$$\lim_{k\to\infty} \sqrt[k]{\|\mathbf{y}\left(k\right)\|}$$

exists and is equal to the modulus of one of the eigenvalues of the matrix A.

Lemma 6.1. *The recurrence system* (16) *is of Poincaré type if and only if*

$$\alpha_n \geqslant \alpha_j \qquad j = 0, 1, \dots, n. \tag{18}$$

Proof. Clearly, a necessary and sufficient condition for (16) being of Poincaré type is that

$$\deg q_\ell\left(k\right) \leqslant \deg q_\alpha\left(k\right), \quad \ell = -n \dots, \alpha - 1.$$

We will show that this condition is equivalent to (18).

Recall the definition (12). The highest power of k in any $q_\ell(k)$ is determined by the maximal index $j = 0, \dots, n$ for which $a_{i,j} \neq 0$ and $i - j = \ell$. Consider $\ell = \alpha_n = d_n - n$, then obviously since $a_{d_n,n} \neq 0$ we have $\deg q_{\alpha_n}(k) = n$.

1. Now let's assume that for some $s < n$ we have $\alpha_s > \alpha_n$, i.e. $d_s - s > d_n - n$, and consider the polynomial $q_{\alpha_s}(k)$:

$$q_{\alpha_s}(k) = \sum_{j=0}^{n} (-1)^j a_{j+\alpha_s,j} (k + j + \alpha_s)_j .$$

   By assumption, $\alpha_s + n > d_n$, and therefore in this case $\deg q_{\alpha_s}(k) < n$. Thus if $\alpha_s > \alpha_n$ for some $s < n$, we have $\deg q_{\alpha_n} > \deg q_{\alpha_s}$. In particular, $\alpha \geqslant \alpha_s > \alpha_n$ and therefore $\deg q_\alpha < \deg q_{\alpha_n}$. This proves one direction.
2. In the other direction, assume that $\alpha = \max_j \alpha_j = \alpha_n$. Clearly $\deg q_\alpha = \deg q_{\alpha_n} = n$, but on the other hand it is always true that $\deg q_\ell \leqslant n$.

This concludes the proof. □

Remark 6.1. The condition (18) in fact means that the point $z = \infty$ is at most a regular singularity of the operator $\mathfrak{D}$.

So in the remainder of the section we assume that (18) is satisfied and $n > 0$. The constant part of the system (16) is the matrix

$$A = \begin{bmatrix} \begin{matrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots \\ -\beta_{-n} & -\beta_{-n+1} & \dots & & -\beta_{-n+d_n-1} \end{matrix} & \mathbf{0}^{d_n \times \tau} \\ \mathbf{0}^{\tau \times d_n} & \begin{matrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots \\ b_\tau & b_{\tau-1} & \dots & & b_1 \end{matrix} \end{bmatrix},$$

where

$$\beta_{-n+s} \overset{\text{def}}{=} \lim_{k\to\infty} \frac{q_{-n+s}(k)}{q_{\alpha_n}(k)} = \frac{a_{s,n}}{a_{d_n,n}}.$$

Proposition 6.2. *The set Z_A of the eigenvalues of the matrix A is precisely the union of the roots of $p_n(x)$ (i.e. the singular points of the operator $\mathfrak{D}$) and the jump points $\{x_i\}_{i=0}^{p+1}$.*

Proof. This is immediate, since $A = \begin{bmatrix} C & \mathbf{0} \\ \mathbf{0} & D \end{bmatrix}$, where C is the companion matrix of $p_n(x)$ and D is the companion matrix of the characteristic polynomial $\Theta_{\mathcal{S}}(z)$ as defined in (14).

In [2] we study the following question: *how many first moments* $\{m_k\}$ *can vanish for a nonzero* $g \in \mathcal{PD}(\mathfrak{D}, p)$? In particular, we prove the following result.

Theorem 6.3 ([2]). *Let the operator* $\mathfrak{D}$ *be of Fuchsian type (i.e. having only regular singular points, possibly including* ∞*). In particular,* $\mathfrak{D}$ *satisfies the condition* (18)*. Let* $g \in \mathcal{PD}(\mathfrak{D}, p)$*.*

1. *If there is at least one discontinuity point* $\xi \in [a, b]$ *of* g *at which the operator* $\mathfrak{D}$ *is nonsingular, i.e.* $p_n(\xi) \neq 0$*, then vanishing of the first* $\tau + d_n - n$ *moments* $\{m_k\}_{k=0}^{\tau+d_n-n-1}$ *of* g *implies* $g \equiv 0$*.*
2. *Otherwise, let* $\Lambda(\mathfrak{D})$ *denote the largest positive integer characteristic exponent of* $\mathfrak{D}$ *at the point* ∞*. In fact, the indicial equation of* $\mathfrak{D}$ *at* ∞ *is* $q_\alpha(k) = 0$*. Then the vanishing of the first* $\Lambda(\mathfrak{D}) + 1 + d_n - n$ *moments of* g *implies* $g \equiv 0$*.*

Everything is now in place in order to obtain the following result.

Theorem 6.4. *Let* $g \in \mathcal{PD}(\mathfrak{D}, p)$ *be a not identically zero function, with* $\mathfrak{D}$ *of Fuchsian type. Then the Stieltjes transform* $S_g(z)$ *is analytic at the origin, and the series*

$$S_g(z) = \sum_{k=0}^{\infty} m_k z^k$$

converges in a disk of radius R *which satisfies*

$$R \geqslant R^* \stackrel{def}{=} \min\left\{\xi^{-1} : \ \xi \in Z_A\right\},$$

where Z_A *is given by Proposition 6.2. Furthermore, for every*

$$N \geqslant \max\{\tau - 1, \Lambda(\mathfrak{D})\} + d_n - n,$$

S_g *satisfies* $(N, R, S(k))$ *Taylor domination, where* $S(k)$ *is given by Lemma 2.1.*

Proof. By Lemma 6.1 and Theorem 6.2 it is clear that

$$\limsup_{k\to\infty} \sqrt[k]{|m_k|} \leqslant \frac{1}{R^*}.$$

By Theorem 6.3, $m_k \neq 0$ for at least some $k = 0, \ldots, N$. Then we apply Corollary 2.1. □

In order to bound the number of zeros of S_g, we would need to estimate the growth of the rational functions $\frac{q_{-n+s}(k)}{q_{\alpha_n}(k)}$, and this can hopefully be done using some general properties of the operator $\mathfrak{D}$. Then we would apply the results of Section 5. We expect that in this way we can single out subclasses

of $\mathcal{PD}$ for which uniform Taylor domination takes place. We plan to carry out this program in a future work.

Remark 6.2. It is possible to obtain Taylor domination for the Stieltjes transforms $S_g(z)$ by another method, based on Remez-type inequalities [3]. We plan to present these results separately.

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